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CONSTITUTIVE ANALYSIS OF LARGE ELASTO-PLASTIC DEFORMATION BASED ON THE MULTIPLICATIVE DECOMPOSITION OF DEFORMATION GRADIENT

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Abstract—By using Lee's (1969, *J. Appl. Mech.* **36**, 1–6) multiplicative decomposition of the deformation gradient into its elastic and plastic part of Hill and Rice's (1973, *SIAM J. Appl. Math.* **25**, 448–461) constitutive framework, explicit and consistent constitutive analysis of large elasto-plastic deformation is given. Both isotropic and anisotropic material behaviour is considered, so that some earlier results come as particular cases of this more general formulation. The relationship with other related work is also given.

1. INTRODUCTION

The constitutive analysis of elasto-plastically deformed materials by using the multiplicative decomposition of the deformation gradient into its elastic and plastic part, introduced by Lee (1969), has since stimulated great interest in both kinematic and kinetic aspects of elasto-plastic theory. However, a proper identification of the elastic and plastic contribution to the total velocity strain, i.e. their relationship with elastic and plastic parts of the deformation gradient, and their rates, have shown not to be an easy task. The reason for this is because the multiplicative decomposition is not kinematically uniquely defined and because, in contrast to pure elasticity theory, the unstressed reference state used to formulate the elastic constitutive relation is not fixed, but rotates due to the plastic part of the deformation gradient. In the case of assumed (persistent) elastic and plastic isotropy of the material, a proper identification of the elastic contribution to the velocity strain was given by Lubarda and Lee (1981). It was shown there that the elastic contribution to the velocity strain is linearly related to the Jaumann rate (with respect to total spin) of the Kirchhoff stress, via the instantaneous elastic compliance tensor [the inverse of their eqn (54)], while the rest of the velocity strain is the plastic contribution, governed by a potential and obeying normality. The rate-type formulation of the theory with strain induced plastic anisotropy (anisotropic hardening) has recently been given in the paper by Agah-Tehrani *et al.* (1987). The elastic isotropy, which remains preserved, was still assumed. The formulation given in this paper is more general as it uses an arbitrary unstressed reference configuration and not a particular one, such as, for example, obtained by destressing without rotation. Of course, final results are independent of the selected intermediate configuration, but this formulation leads to broader interpretations and to some new expressions, such as the relationship between elastic rotation on destressing and the corresponding intermediate spin. A comparison with related work (Mandel, 1973, 1981; Asaro, 1983a,b; Nemat-Nasser, 1982, 1983; Dafalias, 1985, 1987) is made, in particular with respect to the definition of elastic and plastic strain-rates and formulation of corresponding constitutive laws. We also consider the case of elastically anisotropic materials, which was treated little in the literature, by using the multiplicative decomposition and which caused some scepticism concerning the extent of the decomposition utility. The presented analysis can be of interest in establishing an explicit, appropriate constitutive structure capable of reproducing and predicting the complex inelastic behavior. For example, various specific kinematic and kinetic aspects of plastic deformation involving large strains and rotations are extensively studied in the case of large-strain shear reponse, on both a continuum–phenomenological and micromechanical–crystallographic level (Lee *et al.*, 1983; Lubarda, 1988a; Asaro, 1983a,b; Harren *et al.*,

1989, etc.). The onset of instability and prediction of various nonuniform deformation modes was analyzed by Pierce *et al.* (1983). Also, Boyce *et al.* (1989) have recently utilized the multiplicative decomposition in the kinematic analysis of large inelastic deformation of glassy polymers.

2. KINEMATIC AND OTHER PRELIMINARIES

Let $\mathscr{B}_0$ be the initial (undisturbed) configuration of the (polycrystalline metal) body whose material points are specified by the Cartesian coordinates X_i $(i = 1, 2, 3)$. After loading beyond the elastic limit, the body takes on the configuration $\mathscr{B}_t$ (at time t), such that the corresponding deformation gradient is F. By imagining the body to be destressed to zero stress, elastic strains are released and the intermediate (unstressed, relaxed) configuration $\mathscr{P}_t$ is obtained. This configuration differs from the initial configuration by pure plastic deformation. Let F_p denote the corresponding (plastic) part of the total deformation gradient, F_e is the elastic part of F, which corresponds to (elastic) stressing from $\mathscr{P}_t$ to $\mathscr{B}_t$ and the multiplicative decomposition $F = F_eF_p$ holds (Lee, 1969). In Lee's work the nonuniqueness of the intermediate configuration (due to possible superimposed rigid body rotation) was eliminated by defining the elastic "deformation gradient" F_e, to be rotation free and hence given by a symmetric matrix. For the sake of generality, however, we assume here arbitrary rotation on destressing, so that by polar decomposition $F_e = V_eR_e$. On superimposing the time-dependent, rigid-body rotation Q in the current configuration $\mathscr{B}_t$, the deformation gradient F changes to $F^* = QF$, while the elastic and plastic parts F_e and F_p change to $F_e^* = QF_eQ^T$ and $F_p^* = \hat{Q}F_p$ (T standing for transpose); where the orthogonal (rotation) tensor $\hat{Q}$ depends on selected rotation R_e during elastic destressing, i.e. on the selected intermediate configuration. (For example, if $R_e \equiv I$, then $\hat{Q} \equiv Q$; if R_e is such that, for example, the spin of intermediate configuration is zero, then $\hat{Q} \equiv I$.) The elastic stretch V_e and rotation R_e, change accordingly to $V_e^* = QV_eQ^T$ and $R_e^* = QR_e\hat{Q}^T$, while the left and right Cauchy–Green deformation tensors $B_e = F_eF_e^T$ and $C_e = F_e^TF_e = R_e^TB_eR_e$ become $B_e^* = QB_eQ^T$ and $C_e^* = \hat{Q}C_e\hat{Q}^T$. We now introduce two spins (Ω and Ω_p), firstly associated with the current and secondly with the intermediate state, such that under introduced frame changes they behave according to:

$$\begin{aligned}\Omega^* &= \dot{Q}Q^{-1} + Q\Omega Q^T \\ \Omega_p^* &= \dot{\hat{Q}}\hat{Q}^{-1} + \hat{Q}\Omega_p\hat{Q}^T,\end{aligned} \tag{1}$$

with the superimposed dot representing the material derivative and -1 for the inverse. The following (Jaumann-type) derivatives associated with the spin Ω and Ω_p, can then be defined together with the rules they obey under introduced frame changes:

$$\begin{aligned}
&\overset{\circ}{F}_e = \dot{F}_e - \Omega F_e + F_e\Omega_p, && (\overset{\circ}{F}_e)^* = Q\overset{\circ}{F}_e\hat{Q}^T \\
&\overset{\circ}{F}_p = \dot{F}_p - \Omega_p F_p, && (\overset{\circ}{F}_p)^* = \hat{Q}\overset{\circ}{F}_p \\
&\overset{\circ}{F} = \dot{F} - \Omega F, && (\overset{\circ}{F})^* = Q\overset{\circ}{F} \\
&\overset{\circ}{V}_e = \dot{V}_e - \Omega V_e + V_e\Omega, && (\overset{\circ}{V}_e)^* = Q\overset{\circ}{V}_eQ^T \\
&\overset{\circ}{R}_e = \dot{R}_e - \Omega R_e + R_e\Omega_p, && (\overset{\circ}{R}_e)^* = Q\overset{\circ}{R}_e\hat{Q}^T \\
&\overset{\circ}{B}_e = \dot{B}_e - \Omega B_e + B_e\Omega, && (\overset{\circ}{B}_e)^* = Q\overset{\circ}{B}_eQ^T \\
&\overset{\circ}{C}_e = \dot{C}_e - \Omega_p C_e + C_e\Omega_p, && (\overset{\circ}{C}_e)^* = \hat{Q}\overset{\circ}{C}_e\hat{Q}_T.
\end{aligned} \tag{2}_{1-7}$$

For example, $\overset{\circ}{F}_e$ gives the change of elastic deformation gradient F_e observed in the coordinate systems that rotate with spin Ω in the current, and Ω_p in the intermediate configuration. To be compatible with introduced frame change rules, Ω_p depends on the selected intermediate configuration: for example, if $R_e \equiv I$, then $\Omega_p \equiv \Omega$; if R_e is such that the spin of the intermediate configuration is zero ($\mathscr{W}_p = 0$), then $\Omega_p \equiv 0$. If the intermediate configuration is isoclinic (in the sense of Mandel, 1973), again $\Omega_p \equiv 0$. A more specific interpretation and explicit representation of the spin Ω will be given in Section 3 of the paper.

Next, by using multiplicative decomposition, the velocity gradient can be expressed as

$$L = \dot{F}_e F_e^{-1} + F_e(\mathscr{D}_p + \mathscr{W}_p)F_e^{-1}, \tag{3}$$

where $\mathscr{D}_p$ and $\mathscr{W}_p$ are the plastic velocity strain and spin of the intermediate configuration, i.e. symmetric and antisymmetric part of $\dot{F}_p F_p^{-1}$, respectively. Clearly, $\mathscr{D}_p^* = \hat{Q}\mathscr{D}_p\hat{Q}^T$ and $\mathscr{W}_p^* = \dot{\hat{Q}}\hat{Q}^{-1} + \hat{Q}\mathscr{W}_p\hat{Q}^T$. In view of $L = D + W$ (W being the total spin), by taking the symmetric part of eqn (3), we obtain the velocity strain in the current configuration

$$D = (\dot{F}_e F_e^{-1})_s + [F_e(\mathscr{D}_p + \mathscr{W}_p)F_e^{-1}]_s. \tag{4}$$

Identification (separation) of the elastic and plastic contribution in the right-hand side of eqn (4) has caused many disagreements in the literature. In the next section we elaborate on this issue by using Hill and Rice's general constitutive framework.

3. ELASTIC STRAIN-RATE

Following Hill and Rice (1973), let E be any objective, symmetric strain tensor and T its work conjugate (symmetric) stress, such that the Pfaffian T: dE is the increment of work per unit volume in the reference state from where E is measured (: denotes the trace). Further, let $\mathscr{L}_e$ be the corresponding instantaneous elastic moduli tensor, such that $\mathscr{L}_e$: dE is the stress increment that would result if the response on the arbitrary strain variation dE were purely elastic. With $\mathscr{M}_e = \mathscr{L}_e^{-1}$ being the instantaneous elastic compliance tensor, $\mathscr{M}_e$: dT is the strain increment that would result from a purely elastic response corresponding to a stress increment dT, and substruction from the actual strain increment dE gives the plastic part

$$\mathrm{d}_p E = \mathrm{d}E - \mathscr{M}_e : \mathrm{d}T, \tag{5}$$

which is the residual strain increment in an infinitesimal loading–unloading cycle of stress T. This quantity can be shown to be governed by the plastic potential, i.e. codirectional with the outward normal to a locally smooth yield surface in stress T space. For example, by identifying E as the Lagrange strain, T is the symmetric Piola–Kirchhoff stress and by taking the reference state coincident with the current state, we have: dE = D dt and dT = $[\overset{\square}{\sigma} + (\mathrm{tr}\, D)\sigma]\,\mathrm{d}t$, where superimposed $\square$ denotes the convected derivative, i.e. $\overset{\square}{\sigma} = \dot{\sigma} - L\sigma - \sigma L^T = \mathring{\sigma} - D\sigma - \sigma D$, $\mathring{\sigma}$ being the Jaumann derivative of Cauchy stress with respect to total spin W. If E is the logarithmic strain (Hill, 1978), then with the current state as a reference: dE = D dt and dT = $[\mathring{\sigma} + (\mathrm{tr}\, D)\sigma]\,\mathrm{d}t = \mathring{\tau}\,\mathrm{d}t$ [Jaumann increment of the Kirchhoff stress $\tau = (\det F)\sigma$ at $F \equiv I$], so that $\mathrm{d}_p E = (D - \mathscr{M}_e : \mathring{\tau})\,\mathrm{d}t$, i.e.

$$D_p = D - \mathscr{M}_e : \mathring{\tau}. \tag{6}$$

Of course, $\mathscr{M}_e$ changes with different choice of conjugate variables, i.e. different objective stress–rates, and what appears to be the elastic and plastic strain-rates (D_e and $D_p = D - D_e$) depends on the chosen stress-rate. We shall most conveniently work with

expression (6), i.e. define elastic strain-rate by $D_e = \mathcal{M}_e : \mathring{\tau}$ and express the elastic compliance $\mathcal{M}_e$ by using the finite elasticity law, which in the case of (persistent) elastic isotropy has the form

$$(\det V_e)\sigma = 2V_e \frac{\partial w(B_e)}{\partial B_e} V_e; \tag{7}$$

the strain energy (per unit of unstressed volume) w being an isotropic function of B_e. As we shall see, even if intended applications are towards problems with small elastic components of strain, for consistent development it is essential to start the rate-type analysis from this, finite elasticity law. Indeed, by applying the Jaumann-type derivative introduced in Section 2 to both sides of eqn (7), we have

$$\mathring{\tau} = (\mathring{V}_e V_e^{-1})\sigma + \sigma(V_e^{-1}\mathring{V}_e) + \frac{2}{\det V_e} V_e \left[\frac{\partial^2 w(B_e)}{\partial B_e \otimes \partial B_e} : \mathring{B}_e \right] V_e, \tag{8}$$

where $\mathring{\tau} = \mathring{\sigma} + (\operatorname{tr} D)\sigma$, while $\otimes$ denotes the tensor product. The plastic incompressibility assumption has been used in arriving from eqn (7) to eqn (8), i.e. $(\det V_e)^{\cdot} = (\det F)^{\cdot} = (\det V_e)(\operatorname{tr} D)$. But, from eqn $(2)_4$, symmetric and antisymmetric parts of the matrix $\mathring{V}_e V_e^{-1}$ are:

$$(\mathring{V}_e V_e^{-1})_s = (\dot{V}_e V_e^{-1} + V_e \Omega V_e^{-1})_s \equiv \tfrac{1}{2} V_e^{-1} \mathring{B}_e V_e^{-1} \tag{9}$$

and

$$(\mathring{V}_e V_e^{-1})_a = (\dot{V}_e V_e^{-1} + V_e \Omega V_e^{-1})_a - \Omega \equiv \omega - \Omega, \tag{10}$$

with the obvious representation of the spin, ω. Therefore, by defining Ω to be such that $\omega \equiv W$ (total spin), expression (9) gives the elastic strain-rate

$$(\mathring{V}_e V_e^{-1})_s \equiv D_e, \tag{11}$$

since, then, substitution of (9) and (10) into eqn (8) yields

$$\mathring{\tau} = D_e\sigma + \sigma D_e + \frac{4}{\det V_e} V_e \left[\frac{\partial^2 w(B_e)}{\partial B_e \otimes \partial B_e} : (V_e D_e V_e) \right] V_e, \tag{12}$$

i.e. the needed relationship between $\mathring{\tau}$ and elastic strain-rate D_e

$$\mathring{\tau} = \mathscr{L}_e : D_e, \quad D_e = \mathscr{M}_e : \mathring{\tau} \quad (\mathscr{M}_e = \mathscr{L}_e^{-1}). \tag{13}$$

In the component form, the elastic moduli tensor $\mathscr{L}_e$ is

$$\mathscr{L}^e_{ijkl} = \tfrac{1}{2}(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk}) + \frac{4}{\det V_e} V^e_{im} V^e_{jn} \frac{\partial^2 w(B_e)}{\partial B^e_{mn} \partial B^e_{pq}} V^e_{pk} V^e_{ql}, \tag{14}$$

with the obvious symmetry and reciprocity properties (δ being the Kronecker delta).

Clearly, D_e is objective and independent of the selected intermediate configuration, i.e. rotation $\hat{Q}$. Indeed, by using eqns $(2)_4$ and $(2)_6$, from (9) it follows that

$$D_e^* = \tfrac{1}{2}(V_e^{-1})^*(\overset{\diamond}{R}_e)^*(V_e^{-1})^* \equiv QD_eQ^T, \tag{15}$$

which is also clear from (13), since neither $\overset{\diamond}{\tau}$ not $\mathscr{L}_e$ depend on $\hat{Q}$. If the elastic component of the strain is small ($V_e \approx I$), the elastic moduli tensor reduces to

$$\mathscr{L}^e_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{16}$$

as the acting stress is far smaller than the (Lamé) elastic moduli λ and μ.

Finally, in this section, we give the explicit expression for the spin Ω. First, we observe the identity

$$V_e^{-1}(\overset{\diamond}{V}_eV_e^{-1}) = (\overset{\diamond}{V}_eV_e^{-1})^TV_e^{-1}, \tag{17}$$

hence, since the symmetric part of $(\overset{\diamond}{V}_eV_e^{-1})$ is D_e, the antisymmetric part will be

$$(\overset{\diamond}{V}_eV_e^{-1})_a = Z_1H_1 + H_1Z_1 - (\operatorname{tr} Z_1)H_1, \tag{18}$$

where $Z_1 = [(\operatorname{tr} V_e^{-1})I - V_e^{-1}]^{-1}$ and $H_1 = V_e^{-1}D_e - D_eV_e^{-1}$. [Indeed, eqn (17) is of the form $AX = X^TA$; A being the symmetric matrix, which for a given symmetric part of X, can be solved for X; see details, for example, in Agah-Tehrani *et al.*, 1987.] Combining (18) with (10), we then obtain

$$\Omega = W - (Z_1H_1 + H_1Z_1) + (\operatorname{tr} Z_1)H_1. \tag{19}$$

Note that $\Omega^* = \dot{Q}Q^{-1} + Q\Omega Q^T$, as it should be, since Ω is independent of the selected intermediate configuration. If the elastic component of the strain is small, $\Omega \approx W$.

4. PLASTIC STRAIN-RATE

After substructing the elastic contribution $D_e = \mathscr{M}_e : \overset{\diamond}{\tau}$ from the total velocity strain, we obtain the plastic strain rate D_p, which obeys the normality rule. Indeed, by applying the Il'yushin postulate to certain limiting types of infinitesimal strain cycles, Hill (1968) has shown that the inequality $\delta T : \mathrm{d}_pE < 0$ in conjugate variables (E, T), holds for any stress increment δT emanating from the same T and directed inside the yield surface. Hence d_pE is codirectional with the outward normal to the yield surface in T space. By taking E to be the logarithmic strain and by choosing the current state as a reference, the above inequality gives $\overset{\diamond}{\tau} : D_p > 0$, i.e. a positive scalar product of the plastic strain-rate D_p and corresponding (associated) Jaumann rate $\overset{\diamond}{\tau}$. The normality of D_p can then be expressed as

$$D_p = \dot{\lambda}\frac{\partial f}{\partial \sigma}, \tag{20}$$

where $\dot{\lambda}$ is a loading index and f a yield function, dependent on the current stress state and some measures of prior plastic deformation. As an illustration of the procedure for establishing a constitutive expression for the plastic strain-rate, consider a model of combined isotropic–kinematic hardening with the Mises-type yield condition

$$f(T_0, \alpha_0, \bar{\sigma})_{F=I} = 0: \quad [\tfrac{1}{2}(T_0' - \alpha_0) : (T_0' - \alpha_0)]_{F=I} - \tfrac{1}{3}\bar{\sigma}^2 = 0, \tag{21}$$

where $\bar{\sigma}$ is the equivalent (yield) stress, the prime denotes the deviatoric part, while T_0 is the work conjugate to logarithmic strain ($E_0 = \ln U$), as discussed in Section 3, which can be expressed (Hill, 1968, 1978) as

$$T_0 = (\det F)R^{\mathrm{T}}\sigma R + O(E^2), \tag{22}$$

R being the rotation tensor in the polar decomposition $F = RU$. If the current state is taken as a reference ($F = I$), then $T_0 = \sigma$, but $\dot{T}_0 = \mathring{\tau}$. Similarly, via the analogous formula to (22), we define α_0 back stress in terms of the (deviatoric) back stress α (in the Cauchy stress space), so that at $F = I$: $\alpha_0 = \alpha$ and $\dot{\alpha}_0 = \mathring{\alpha} + (\mathrm{tr}\, D)\alpha$, which will subsequently be denoted by $\mathring{\beta}$. The consistency condition ($\dot{f} = 0$) accordingly becomes

$$(\mathring{\tau}' - \mathring{\beta}) : (\sigma' - \alpha) - \tfrac{2}{3}\bar{\sigma}\dot{\bar{\sigma}} = 0. \tag{23}$$

To proceed further, we introduce the evolution law for the back stress in the form (Fardshisheh and Onat, 1974; Agah-Tehrani *et al.*, 1987)

$$\mathring{\beta} = A(\alpha, D_{\mathrm{p}}), \tag{24}$$

where A is an isotropic symmetric tensor function of both α and D_{p}. By using the representation theorem, A can be represented in terms of certain basic functions (Spencer, 1971), which, in view of the rate independence, can be written as

$$A(\alpha, D_{\mathrm{p}}) = \mathscr{A}_1(\alpha) : D_{\mathrm{p}} + A_0(\alpha)(\tfrac{2}{3}D_{\mathrm{p}} : D_{\mathrm{p}})^{1/2}, \tag{25}$$

A_0 and $\mathscr{A}_1$ being second and fourth-order tensor functions of the back stress and its invariants. Introducing then, the strain-hardening hypothesis:

$$\bar{\sigma} = \bar{\sigma}(\bar{E}_{\mathrm{p}}), \quad \bar{E}_{\mathrm{p}} = \int_0^t (\tfrac{2}{3}D_{\mathrm{p}} : D_{\mathrm{p}})^{1/2}\, \mathrm{d}t, \tag{26}$$

substitution into the consistency condition (23), gives, in view of (20) and (24), the following expression for the loading index

$$\lambda = \frac{1}{h}(\sigma' - \alpha) : \mathring{\tau}, \tag{27}$$

where

$$h = \tfrac{4}{9}\bar{\sigma}^2 \frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{E}_{\mathrm{p}}} + [\mathscr{A}_1(\alpha) : (\sigma' - \alpha) + \tfrac{2}{3}\bar{\sigma}A_0(\alpha)] : (\sigma' - \alpha). \tag{28}$$

Therefore, the plastic strain-rate is

$$D_{\mathrm{p}} = \mathscr{M}_{\mathrm{p}} : \mathring{\tau}, \quad \mathscr{M}_{\mathrm{p}} = \frac{1}{h}(\sigma' - \alpha) \otimes (\sigma' - \alpha), \tag{29}$$

with the obvious symmetry and reciprocity properties of the plastic compliance tensor $\mathscr{M}_{\mathrm{p}}$. This, combined with expression (13) for the elastic strain-rate, gives the well known structure of the rate-type elasto-plastic constitutive law $D = \mathscr{M} : \mathring{\tau}$, $\mathscr{M} = \mathscr{M}_{\mathrm{e}} + \mathscr{M}_{\mathrm{p}}$ being the instantaneous elastic–plastic compliance (associated with the Jaumann stress rate $\mathring{\tau}$). The usual inversion provides the (self-adjoint) elastic–plastic moduli $\mathscr{L}$ and the constitutive structure $\mathring{\tau} = \mathscr{L} : D$.

5. RELATIONSHIP BETWEEN PLASTIC STRAIN-RATES IN THE CURRENT AND INTERMEDIATE CONFIGURATIONS

It is of interest to derive the relationship between the plastic strain-rate D_{p} in the current configuration and the plastic strain-rate $\mathscr{D}_{\mathrm{p}}$ in the intermediate configuration. To this goal, from eqn (6), in view of $F_{\mathrm{e}} = V_{\mathrm{e}}R_{\mathrm{e}}$, we have

$$D_e+D_p+W=\dot{V}_eV_e^{-1}+V_e[\dot{R}_eR_e^{-1}+R_e(\mathscr{D}_p+\mathscr{W}_p)R_e^T]V_e^{-1}, \tag{30}$$

where from, on using (9) and (10) to eliminate $\dot{V}_eV_e^{-1}$,

$$D_p=V_e[\dot{R}_eR_e^{-1}-\Omega+R_e(\mathscr{D}_p+\mathscr{W}_p)R_e^T]V_e^{-1}. \tag{31}$$

Hence

$$R_e\mathscr{D}_pR_e^T=(V_e^{-1}D_pV_e)_s\equiv\tfrac{1}{2}V_e^{-1}(B_eD_p+D_pB_e)V_e^{-1} \tag{32}$$

and

$$\dot{R}_eR_e^{-1}-\Omega+R_e\mathscr{W}_pR_e^T=(V_e^{-1}D_pV_e)_a. \tag{33}$$

However, the identity

$$B_e(V_e^{-1}D_pV_e)=(V_e^{-1}D_pV_e)^TB_e \tag{34}$$

holds, and by using expression (32) as in Section 3, we obtain the antisymmetric part

$$(V_e^{-1}D_pV_e)_a=Z_2H_2+H_2Z_2-(\operatorname{tr}Z_2)H_2, \tag{35}$$

where

$$\begin{aligned}Z_2&=[(\operatorname{tr}B_e)I-B_e]^{-1}\\ H_2&=B_2(R_e\mathscr{D}_pR_e^T)-(R_e\mathscr{D}_pR_e^T)B_e\equiv[V_e(B_eD_p+D_pB_e)V_e^{-1}]_a.\end{aligned} \tag{36}$$

Substitution of eqns (33), (35) and (36) into eqn (31) then gives the relationship between D_p and $\mathscr{D}_p$

$$D_p=F_e[\mathscr{D}_p+Z_0(C_e\mathscr{D}_p-\mathscr{D}_pC_e)+(C_e\mathscr{D}_p-\mathscr{D}_pC_e)Z_0-(\operatorname{tr}Z_0)(C_e\mathscr{D}_p-\mathscr{D}_pC_e)]F_e^{-1}, \tag{37}$$

with $Z_0=[(\operatorname{tr}C_e)I-C_e]^{-1}$. Since $F_e^*=QF_e\hat{Q}^T$, $C_e^*=\hat{Q}C_e\hat{Q}^T$, $D_p^*=\hat{Q}\mathscr{D}_p\hat{Q}^T$ and $Z_0=\hat{Q}Z_0\hat{Q}^T$, we have $D_p^*=QD_pQ^T$, i.e. D_p is independent of $\hat{Q}$ (as, of course, it must be since we have already seen that D and D_e are both independent of $\hat{Q}$). In the case of isotropic hardening, the principal directions of stress and plastic strain-rate coincide [following from eqn (20)], and as with isotropic elasticity, V_e also has principal directions parallel to stress; the expression given by (35) is identically equal to zero and, therefore, $D_p=R_e\mathscr{D}_pR_e^T$.

We now derive the relationship between elastic rotation on destressing and the corresponding intermediate spin. Substitution of (19) for the spin Ω and (35) for the spin $(V_e^{-1}D_pV_e)_a$ into the relationship (33), gives

$$\dot{R}_eR_e^{-1}+R_e\mathscr{W}_pR_e^T=W-(Z_1H_1+H_1Z_1)+(Z_2H_2+H_2Z_2)+(\operatorname{tr}Z_1)H_1-(\operatorname{tr}Z_2)H_2, \tag{38}$$

which expresses the spin $\dot{R}_eR_e^{-1}+R_e\mathscr{W}_pR_e^T$ in terms of the total spin W, elastic deformation V_e and elastic and plastic strain-rates D_e and D_p [see expressions (36) for Z_2 and H_2, and similar ones for Z_1 and H_1, following eqn (18)]. For example, if the intermediate configuration is obtained by destressing without rotation ($R_e\equiv I$, $\dot{R}_e\equiv 0$), (38) gives the corresponding spin $\mathscr{W}_p$. If destressing is defined such that $\mathscr{W}_p\equiv 0$, (38) gives $\dot{R}_eR_e^{-1}$, i.e. on integrating, the corresponding elastic rotation R_e. In the case of isotropic hardening, (38) is simplified, since then $H_2\equiv 0$.

6. SOME PARTICULAR CASES AND RELATIONSHIP WITH PREVIOUS RESULTS

(i) As a first particular case of presented formulation, consider the case of the intermediate configuration obtained by destressing without rotation, as used in the work by Lee and his coworkers. Then $F_e = V_e$, $R_e = I$, $C_e = B_e = V_e^2$ and eqn (37) becomes

$$D_p = V_e[\mathscr{D}_p + Z_2(B_e\mathscr{D}_p - \mathscr{D}_pB_e) + (B_e\mathscr{D}_p - \mathscr{D}_pB_e)Z_2 - (\operatorname{tr} Z_2)(B_e\mathscr{D}_p - \mathscr{D}_pB_e)]V_e^{-1}, \tag{39}$$

which, with somewhat different notation, coincides with expressions (30) and (31) of Agah-Tehrani *et al.* (1987). If hardening is isotropic, $D_p = \mathscr{D}_p$, as utilized by Lubarda and Lee (1981). From eqn (38), on the other hand, we get the plastic spin of the intermediate configuration

$$\mathscr{W}_p = W - (Z_1H_1 + H_1Z_1) + (Z_2H_2 + H_2Z_2) + (\operatorname{tr} Z_1)H_1 - (\operatorname{tr} Z_2)H_2, \tag{40}$$

explicitly in terms of W, V_e, D_e and D_p. Note also that from eqn (33) we have $\Omega = \mathscr{W}_p - (V_e^{-1}D_pV_e)_a$. If hardening is isotropic, $\Omega = \mathscr{W}_p$, which agrees with Lubarda and Lee (1981), who used the Jaumann derivative with respect to spin $\mathscr{W}_p$ to define elastic strain-rate [eqn (21) of the cited paper].

(ii) Consider next the intermediate configuration obtained by the destressing program such that always $\mathscr{W}_p = 0$. Equation (38) then gives the corresponding rotation R_e, via the spin

$$\dot{R}_eR_e^{-1} = W - (Z_1H_1 + H_1Z_1) + (Z_2H_2 + H_2Z_2) + (\operatorname{tr} Z_1)H_1 - (\operatorname{tr} Z_2)H_2. \tag{41}$$

Again, this is known in terms of W, V_e, D_e and D_p. If hardening is isotropic, $\dot{R}_eR_e^{-1} = \Omega$.

(iii) Particularly interesting, and in many aspects revealing, is the special case of the intermediate configuration defined by the destressing program such that

$$[F_e(\mathscr{D}_p + \mathscr{W}_p)F_e^{-1}]_a = 0. \tag{42}$$

From eqn (31) then follows $[V_e(\dot{R}_eR_e^{-1} - \Omega)V_e^{-1}]_a = 0$, which (by simple matrix arguments) necessarily implies

$$\dot{R}_eR_e^{-1} = \Omega = W - (Z_1H_1 + H_1Z_1) + (\operatorname{tr} Z_1)H_1, \tag{43}$$

while eqn (38) gives the spin of the intermediate state

$$\mathscr{W}_p = R_e^T[(Z_2H_2 + H_2Z_2) - (\operatorname{tr} Z_2)H_2]R_e. \tag{44}$$

In the case of isotropic hardening ($H_2 = 0$), this is identically equal to zero ($\mathscr{W}_p = 0$), which coincides with Mandel's (1973) result. Observe, however, that in view of (43), the elastic strain-rate (11) becomes

$$D_e = [\dot{V}_eV_e^{-1} + V_e(\dot{R}_eR_e^{-1})V_e^{-1}]_s = (\dot{F}_eF_e^{-1})_s, \tag{45}$$

while (31) reduces to

$$D_p = F_e(\mathscr{D}_p + \mathscr{W}_p)F_e^{-1}. \tag{46}$$

Equations (43)–(46), therefore, show that if elastic strain-rate is defined as the symmetric part of $\dot{F}_eF_e^{-1}$, condition (42) necessarily holds. This seems to be overlooked in much of the previous work on this subject.

It is, perhaps, instructive to analyze this particular case a little further. From eqn (3), in view of (42), we first have:

$$D = (\dot{F}_e F_e^{-1})_s + F_e(\mathscr{D}_p + \mathscr{W}_p)F_e^{-1} \tag{47}$$

and

$$W = (\dot{F}_e F_e^{-1})_a, \tag{48}$$

while differentiation of the finite elasticity law

$$(\det F_e)\sigma = 2F_e \frac{\partial w(C_e)}{\partial C_e} F_e^T, \tag{49}$$

w being (for isotropic elasticity) an isotropic function of C_e, gives

$$\dot{\tau} = (\dot{F}_e F_e^{-1})\sigma + \sigma(F_e^{-T}\dot{F}_e^T) + \frac{2}{\det F_e} F_e\left[\frac{\partial^2 w(C_e)}{\partial C_e \otimes \partial C_e} : \dot{C}_e\right]F_e^T. \tag{50}$$

From (50), in view of (48), it is clear that $(\dot{F}_e F_e^{-1})_s$ is indeed the elastic strain-rate D_e, since then

$$\dot{\tau} = D_e\sigma + \sigma D_e + \frac{4}{\det F_e} F_e\left[\frac{\partial^2 w(C_e)}{\partial C_e \otimes \partial C_e} : (F_e^T D_e F_e)\right]F_e^T \equiv \mathscr{L}_e : D_e, \tag{51}$$

as it should be [see eqns (12) and (13)]. Note also the isotropic elasticity identity

$$F_e\left[\frac{\partial^2 w(C_e)}{\partial C_e \otimes \partial C_e} : (F_e^T D_e F_e)\right]F_e^T \equiv V_e\left[\frac{\partial^2 w(B_e)}{\partial B_e \otimes \partial B_e} : (V_e D_e V_e)\right]V_e. \tag{52}$$

For any other choice of intermediate configuration [not defined by (42)], the symmetric part of $\dot{F}_e F_e^{-1}$ is not all, but just a portion of the elastic strain-rate D_e. See also further discussion on this in the next section, within the context of anisotropic elasticity.

7. ELASTICALLY-ANISOTROPIC MATERIALS

Let us now consider elastically-anisotropic (say, orthotropic) materials and let $\mathbf{a}_i^0$ ($i = 1, 2, 3$) define the axes of anisotropy with respect to which the strain energy has the representation $w = w(C_e)$. We can also assume further, that the material in its unstressed configuration has the same type of elastic anisotropy as in its initial state, so that the axes $\mathbf{a}_i^0$ have just rotated to axes $\mathbf{a}_i^+$ in the intermediate state, i.e. $\mathbf{a}_i^+ = \mathscr{R}\mathbf{a}_i^0$ ($\mathscr{R}$ being the orthogonal tensor). The strain energy can consequently be represented by $w(\mathscr{R}^T C_e \mathscr{R})$. As discussed by Mandel (1973, 1981), in view of the discontinuities of displacements and rotations of elements at the microscale, $\mathscr{R}$ is independent of the (overall) plastic part of deformation gradient. Since final results of the analysis are independent of selected intermediate configurations, in this section we shall consider only the so-called isoclinic intermediate configuration, defined such that $\mathbf{a}_i^+ = \mathbf{a}_i^0$, i.e. $\mathscr{R} = I$. (A more general development of the analysis by using arbitrary intermediate configurations was presented by Lubarda, 1988b.) The corresponding multiplicative decomposition of the deformation gradient is $F = \mathscr{F}_e \mathscr{F}_p$, while the stress response is

$$(\det V_e)\sigma = 2\mathscr{F}_e \frac{\partial w(\mathscr{C}_e)}{\partial \mathscr{C}_e} \mathscr{F}_e^T, \tag{53}$$

where $\mathscr{C}_e = \mathscr{F}_e^T \mathscr{F}_e$. Applying the material derivative to (53), we have

$$\mathring{\tau} = (\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})\sigma + \sigma(\mathscr{F}_e^{-T}\dot{\mathscr{F}}_e^{T}) + \frac{2}{\det V_e}\mathscr{F}_e\left[\frac{\partial^2 w(\mathscr{C}_e)}{\partial\mathscr{C}_e\otimes\partial\mathscr{C}_e}:\dot{\mathscr{C}}_e\right]\mathscr{F}_e^{T}. \tag{54}$$

It then follows that

$$(\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})_s = D_e + \hat{\Delta}_e, \tag{55}$$

since substitution in (54), in view of

$$W = (\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})_a + [\mathscr{F}_e(\dot{\mathscr{F}}_p\mathscr{F}_p^{-1})\mathscr{F}_e^{-1}]_a, \tag{56}$$

gives

$$\mathring{\tau} = \mathscr{L}_e : D_e + \mathscr{L}_e : \hat{\Delta}_e - \hat{W}_p\sigma + \sigma\hat{W}_p, \tag{57}$$

where $\hat{W}_p$ denotes the spin corresponding to the second term on the right-hand side of eqn (56). It is now clear that the additional term $\hat{\Delta}_e$ in eqn (55) has to be such that (57) reduces to $\mathring{\tau} = \mathscr{L}_e : D_e$, hence

$$\hat{\Delta}_e = \mathscr{L}_e^{-1} : (\hat{W}_p\sigma - \sigma\hat{W}_p). \tag{58}$$

This is in accord with the results of Hill and Rice (1973), Hill and Havner (1982) and Asaro (1983a,b). Indeed, by taking the symmetric part of the velocity gradient, expressed via $\mathscr{F}_e$ and $\mathscr{F}_p$ (in the context of crystal plasticity, $\mathscr{F}_e$ would correspond to the lattice contribution to F associated with stretching and rotation of the lattice, whereas $\mathscr{F}_p$ would be due solely to slip), we get

$$D = (\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})_s + [\mathscr{F}_e(\dot{\mathscr{F}}_p\mathscr{F}_p^{-1})\mathscr{F}_e^{-1}]_s, \tag{59}$$

hence, in view of eqn (55) and $D = D_e + D_p$, we have

$$D_p = \hat{D}_p + \mathscr{L}_e^{-1} : (\hat{W}_p\sigma - \sigma\hat{W}_p), \tag{60}$$

where $\hat{D}_p$ is the second term on the right-hand side of eqn (59). As discussed by Hill and Havner (1982) and Asaro (1983a), D_p, which gives the plastic increment of strain in a stress cycle, does not come from the slip deformation ($\hat{D}_p$) alone; there is a further net elastic contribution from the lattice, which is caused by the slip-induced rotation of the lattice relative to the material stress. Nonetheless, it is D_p (and not $\hat{D}_p$) that is governed by the plastic potential. [The distinction between D_p and $\hat{D}_p$ is small and involves $O(\sigma/\mathscr{L}_e)$ terms in comparison to O(1). However, in some applications it is necessary to retain such accuracy (Asaro and Rice, 1977).] In much of the work (for example, Dafalias, 1985, 1987) the elastic strain-rate is defined to be $(\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})_s$, which is the rate of strain associated with the Jaumann stress-rate, corresponding to spin $(\dot{\mathscr{F}}_e\mathscr{F}_e^{-1})_a$. Indeed, Nemat-Nasser (1982, 1983) introduces decompositions $D = D^* + D^p$ and $W = W^* + W^p$, defining the elastic contribution to the strain-rate by

$$\mathring{\tau} = \mathscr{L}_e : D^*, \quad \mathring{\tau} = \dot{\tau} - W^*\sigma + \sigma W^*. \tag{61}$$

It then directly follows that:

$$\mathring{\tau} = \mathscr{L}_e : D - (\mathscr{L}_e : D^p + W^p\sigma - \sigma W^p). \tag{62}$$

Since, in our context $D^* + W^*$ can be identified as $\dot{\mathscr{F}}_e\mathscr{F}_e^{-1}$, we have $D^p + W^p \equiv \hat{D}_p + \hat{W}_p$ and expression (62), in view of (60), reduces to

$$\mathring{t} = \mathscr{L}_e : (D - D_p), \tag{63}$$

hence, one needs at the phenomenological level to examine the constitutive structure for D_p, rather than D^p. For example, as we have already pointed out, D_p (and not D^p) is governed by a plastic potential. Also, the elastic strain-rate is $D_e = D^* - \hat{\Delta}_e$.

The elastic moduli tensor $\mathscr{L}_e$, appearing in (57) via (54), has the component form

$$\mathscr{L}_e^{ijkl} = \tfrac{1}{2}(\delta_{ik}\sigma_{jl} + \delta_{il}\sigma_{jk} + \sigma_{ik}\delta_{jl} + \sigma_{il}\delta_{jk}) + \frac{4}{\det V_e} \mathscr{F}_{im}^e \mathscr{F}_{jn}^e \frac{\partial^2 w(\mathscr{C}_e)}{\partial \mathscr{C}_{mn}^e \partial \mathscr{C}_{pq}^e} \mathscr{F}_{kp}^e \mathscr{F}_{lq}^e, \tag{64}$$

so that $\mathscr{F}_e = V_e \mathscr{R}_e$ must be determined. If elasticity is infinitesimal ($V_e \approx I$), rotation $\mathscr{R}_e$ is needed. More on this, however, is the subject of a forthcoming paper. If elasticity is isotropic, w becomes an isotropic function of $\mathscr{C}_e$ and (64) reduces to (14).

REFERENCES

Agah-Tehrani, A., Lee, E. H., Mallett, R. L. and Onat, E. T. (1987). The theory of elastic–plastic deformation at finite strain with induced anisotropy modeled as combined isotropic–kinematic hardening. *J. Mech. Phys. Solids* **35**, 519–539.

Asaro, R. J. (1983a). Crystal plasticity. *J. Appl. Mech.* **50**, 921–934.

Asaro, R. J. (1983b). Micromechanics of crystals and polycrystals. In *Advances in Applied Mechanics* (Edited by J. W. Hutchinson and T. Y. Wu), Vol. 23, pp. 1–115. Academic Press, NY.

Asaro, R. J. and Rice, J. R. (1977). Strain localization in ductile single crystals. *J. Mech. Phys. Solids* **20**, 309–338.

Boyce, M. C., Weber, G. G. and Parks, D. M. (1989). On the kinematics of finite strain plasticity. *J. Mech. Phys. Solids* **37**, 647–665.

Dafalias, Y. F. (1985). The plastic spin. *J. Appl. Mech.* **52**, 865–871.

Dafalias, Y. F. (1987). Issues on the constitutive formulation at large elastoplastic deformations, part 1: kinematics. *Acta Mech.* **69**, 119–138.

Fardshisheh, F. and Onat, E. T. (1974). Representation of elastoplastic behaviour by means of state variables. *Proc. Symp. on Foundations of Plasticity*, 89–115. Warsaw, Noordhoff.

Harren, S., Lowe, T. C., Asaro, R. J. and Needleman, A. (1989). Analysis of large-strain shear in rate-dependent face-centred cubic polycrystals: correlation of micro- and macromechanics. *Phil. Trans. R. Soc. Lond.* **A328**, 443–500.

Hill, R. (1968). On constitutive inequalities for simple materials. *J. Mech. Phys. Solids* **16**, 229–242.

Hill, R. (1978). Aspects of invariance in solid mechanics. In *Advances in Applied Mechanics* (Edited by C.-S. Yih), Vol. 18, pp. 1–75. Academic Press, NY.

Hill, R. and Havner, K. S. (1982). Perspectives in the mechanics of elastoplastic crystals. *J. Mech. Phys. Solids* **30**, 5–22.

Hill, R. and Rice, J. R. (1973). Elastic potentials and the structure of inelastic constitutive laws. *SIAM J. Appl. Math.* **25**, 448–461.

Lee, E. H. (1969). Elastic–plastic deformation at finite strains. *J. Appl. Mech.* **36**, 1–6.

Lee, E. H., Mallett, R. L. and Wertheimer, T. B. (1983). Stress analysis for anisotropic hardening in finite-deformation plasticity. *J. Appl. Mech.* **50**, 554–560.

Lubarda, V. A. (1988a). Simple shear of a strain-hardening elastoplastic hollow circular cylinder. *Int. J. Plasticity* **4**, 61–75.

Lubarda, V. A. (1988b). Further contribution to elasto-plastic constitutive analysis based on the multiplicative decomposition of deformation gradient. *ICTAM* 88, Grenoble, p. 128.

Lubarda, V. A. and Lee, E. H. (1981). A correct definition of elastic and plastic deformation and its computational significance. *J. Appl. Mech.* **48**, 35–40.

Mandel, J. (1973). Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques. *Int. J. Solids Structures* **9**, 725–740.

Mandel, J. (1981). Sur la définition de la vitesse de déformation élastique et sa relation avec la vitesse de contrainte. *Int. J. Solids Structures* **17**, 873–878.

Nemat-Nasser, S. (1982). On finite deformation elasto-plasticity. *Int. J. Solids Structures* **18**, 857–872.

Nemat-Nasser, S. (1983). On finite plastic flow of crystalline solids and geomaterials. *J. Appl. Mech.* **50**, 1114–1126.

Pierce, D., Asaro, R. J. and Needleman, A. (1983). Material rate dependence and localized deformation in crystalline solids. *Acta Metall.* **31**, 1951–1976.

Spencer, A. J. M. (1971). Theory of invariants. In *Continuum Physics* (Edited by A. C. Eringen), Vol. 1, pp. 240. Academic Press, NY.